to show how mathematical induction works in studying Diophantine equations.

Example 1. *Prove that for all integers $n \geq 3$, there exist odd positive integers x, y, such that $7x^2 + y^2 = 2^n$.*

(Bulgarian Mathematical Olympiad)

Solution. We will prove that there exist odd positive integers x_n, y_n such that $7x_n^2 + y_n^2 = 2^n$, $n \geq 3$.

For $n = 3$, we have $x_3 = y_3 = 1$. Now suppose that for a given integer $n \geq 3$ we have odd integers x_n, y_n satisfying $7x_n^2 + y_n^2 = 2^n$. We shall exhibit a pair (x_{n+1}, y_{n+1}) of odd positive integers such that $7x_{n+1}^2 + y_{n+1}^2 = 2^{n+1}$. In fact,

$$7\left(\frac{x_n \pm y_n}{2}\right)^2 + \left(\frac{7x_n \mp y_n}{2}\right)^2 = 2(7x_n^2 + y_n^2) = 2^{n+1}.$$

Precisely one of the numbers $\frac{x_n+y_n}{2}$ and $\frac{|x_n-y_n|}{2}$ is odd (since their sum is the larger of x_n and y_n, which is odd). If, for example, $\frac{x_n+y_n}{2}$ is odd, then

$$\frac{7x_n - y_n}{2} = 3x_n + \frac{x_n - y_n}{2}$$

is also odd (as a sum of an odd and an even number); hence in this case we may choose

$$x_{n+1} = \frac{x_n + y_n}{2} \qquad \text{and} \qquad y_{n+1} = \frac{7x_n - y_n}{2}.$$

If $\frac{x_n-y_n}{2}$ is odd, then

$$\frac{7x_n + y_n}{2} = 3x_n + \frac{x_n + y_n}{2},$$

so we can choose

$$x_{n+1} = \frac{|x_n - y_n|}{2} \qquad \text{and} \qquad y_{n+1} = \frac{7x_n + y_n}{2}.$$